S.-T. Yau College Student Mathematics Contests 2017

Oral Exam of Geometry and Topology

Overall Problems

1. Let Σ_g be a compact Riemann surface of genus $g > 1$, $Aut(\Sigma_g)$ be the automorphism group of biholomorphic maps of Σ_g. Let $V = H^0(\Sigma_g, K)$ be the space of holomorphic 1-forms on Σ_g.

(a) Show that the natural group homomorphism

$$\rho : Aut(\Sigma_g) \to GL(V)$$

is injective.

(b) V carries a natural hermitian structure

$$< \omega_1, \omega_2 >= i \int_{\Sigma_g} \omega_1 \wedge \overline{\omega_2}, \quad \omega_i \in V.$$

Show that $\rho(Aut(\Sigma_g))$ lies inside the unitary subgroup.

(c) V carries a natural integral structure from the lattice

$$H^1(\Sigma, \mathbb{Z})(\simeq \mathbb{Z}^{2g}) \subset V.$$

Show that $\rho(Aut(\Sigma_g))$ lies inside $GL(\mathbb{Z}^{2g})$.

(d) Conclude that $Aut(\Sigma_g)$ is a finite group.

2. (a) What is a Killing field on a Riemannian manifold?

(b) Explain why a Killing field on a connected Riemannian manifold is determined by its value and the value of its first derivative at a given point.

(c) Show that the maximal dimension of the space of Killing fields on a three dimensional connected Riemannian manifold is six.

3. (a) Let X be an n-dimensional compact Riemannian manifold. Show that

$$\dim(\mathrm{Isom}(X)) \leq \frac{n(n+1)}{2}.$$

(b) List all possible M when the equality in the above is achieved.